TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 148, April 1970

ANALYTIC SHEAVES OF LOCAL COHOMOLOGY

BY
YUM-TONG SIU

In this paper we are interested in the following problem: Suppose V is an analytic subvariety of a (not necessarily reduced) complex analytic space X, $\mathscr{F}$ is a coherent analytic sheaf on $X-V$, and θ: $X-V \to X$ is the inclusion map. When is $\theta_q(\mathscr{F})$ coherent (where $\theta_q(\mathscr{F})$ is the qth direct image of $\mathscr{F}$ under θ)?

The case $q=0$ is very closely related to the problem of extending $\mathscr{F}$ to a coherent analytic sheaf on X. This problem of extension has already been dealt with in Frisch-Guenot [1], Serre [9], Siu [11]–[14], Thimm [17], and Trautmann [18]–[20]. So, in our investigation we assume that $\mathscr{F}$ admits a coherent analytic extension on all of X.

In reponse to a question of Serre [9, p. 366], Tratumann has obtained the following in [21]:

THEOREM T. *Suppose V is an analytic subvariety of a complex analytic space X, q is a nonnegative integer, and $\mathscr{F}$ is a coherent analytic sheaf on X. Let θ: $X-V \to X$ be the inclusion map. Then a sufficient condition for $\theta_0(\mathscr{F}|X-V), \ldots, \theta_q(\mathscr{F}|X-V)$ to be coherent is: for $x \in V$ there exists an open neighborhood U of x in X such that* codh $(\mathscr{F}|U-V) \geqq \dim_x V+q+2$.

Trautmann's sufficiency condition is in general not necessary, as is shown by the following example. Let $X=C^3$, $V=\{z_1=z_2=0\}$, $q=0$, and $\mathscr{F}=$the analytic ideal-sheaf of $\{z_2=z_3=0\}$. $\theta_0(\mathscr{F}|X-V)=\mathscr{F}$ is coherent on X. However, codh $\mathscr{F}=2$ at $(a, 0, 0)$ for all a. Hence there exists no neighborhood U of $(0, 0, 0)$ such that codh $(\mathscr{F}|U-V) \geqq 3$.

In this paper we derive a *necessary and sufficient* condition for the coherence of $\theta_0(\mathscr{F}|X-V), \ldots, \theta_q(\mathscr{F}|X-V)$. This necessary and sufficient condition is then applied to give an affirmative answer to a question raised by Serre [9, pp. 373–374].

To state the main results, we introduce the following notations: If $\mathscr{F}$ is a coherent analytic sheaf on a complex analytic space X, then $S_k(\mathscr{F})$ denotes the analytic subvariety $\{x \in X \mid \text{codh } \mathscr{F}_x \leqq k\}$, [6, Satz 5, p. 81]. If D is an open subset of X, then $\bar{S}_k(\mathscr{F}|D)$ denotes the topological closure of $S_k(\mathscr{F}|D)$ in X. If V is an analytic subvariety of X, then $\mathscr{H}^k_V(\mathscr{F})$ denotes the sheaf defined by the presheaf $U \mapsto H^k_V(U, \mathscr{F})$, where $H^k_V(U, \mathscr{F})$ is the k-dimensional cohomology group of U with coefficients in $\mathscr{F}$ and supports in V. If X, $\mathscr{F}$, and V are complex algebraic instead of analytic, we use the same notations and $\mathscr{F}^h$ denotes the coherent analytic sheaf canonically associated with $\mathscr{F}$.

Received by the editors April 22, 1969.

The main results are the following two theorems:

THEOREM A. *Suppose V is an analytic subvariety of a complex analytic space $(X, \mathscr{H})$, q is a nonnegative integer, and $\mathscr{F}$ is a coherent analytic sheaf on X. Let $\theta: X-V \to X$ be the inclusion map. Then the following three statements are equivalent*:

(i) $\theta_0(\mathscr{F}|X-V), \ldots, \theta_q(\mathscr{F}|X-V)$ (*or equivalently* $\mathscr{H}_V^0(\mathscr{F}), \ldots, \mathscr{H}_V^{q+1}(\mathscr{F})$) *are coherent on X.*

(ii) *For every $x \in V$, $\theta_0(\mathscr{F}|X-V)_x, \ldots, \theta_q(\mathscr{F}|X-V)_x$ (or equivalently $\mathscr{H}_V^0(\mathscr{F})_x, \ldots, \mathscr{H}_V^{q+1}(\mathscr{F})_x$) are finitely generated over $\mathscr{H}_x$.*

(iii) $\dim V \cap \bar{S}_{k+q+1}(\mathscr{F}|X-V) < k$ *for every* $k \geqq 0$.

THEOREM B. *Suppose V is an algebraic subvariety of a complex algebraic space X, q is a nonnegative integer, and $\mathscr{F}$ is a coherent algebraic sheaf on X. Then $\mathscr{H}_V^0(\mathscr{F}), \ldots, \mathscr{H}_V^{q+1}(\mathscr{F})$ are coherent algebraic sheaves on X if and only if $\mathscr{H}_V^0(\mathscr{F}^h), \ldots, \mathscr{H}_V^{q+1}(\mathscr{F}^h)$ are coherent analytic sheaves on X. If so, the canonical homomorphisms $\mathscr{H}_V^k(\mathscr{F})^h \to \mathscr{H}_V^k(\mathscr{F}^h)$ are isomorphisms for $0 \leqq k \leqq q+1$.*

These results are announced without proofs in [15].

Complex analytic spaces, in this paper, are in the sense of Grauert (i.e., not necessarily reduced). A holomorphic function f on a complex analytic space $(X, \mathscr{H})$ means an element of $\Gamma(X, \mathscr{H})$. We say that f vanishes at a point x of X if f_x is not a unit in $\mathscr{H}_x$.

If n is a nonnegative integer and $\mathscr{F}$ is a coherent analytic sheaf on a complex analytic space X, then $O_{[n]\mathscr{F}}$ denotes the subsheaf of $\mathscr{F}$ defined as follows: for $x \in X$, $(O_{[n]\mathscr{F}})_x = \{s \in \mathscr{F}_x \mid$ for some open neighborhood U of x in X, s is induced by some $t \in \Gamma(U, \mathscr{F})$ satisfying $\dim \operatorname{Supp} t \leqq n\}$. $O_{[n]\mathscr{F}}$ is coherent and $\dim \operatorname{Supp} O_{[n]\mathscr{F}} \leqq n$ (see [10] and [16]). If V is an analytic subvariety of X and $\mathscr{G}$ is a coherent analytic subsheaf of $\mathscr{F}$, then $\mathscr{G}[V]_{\mathscr{F}}$ denotes the subsheaf of $\mathscr{F}$ defined as follows: $(\mathscr{G}[V]_{\mathscr{F}})_x = \{s \in \mathscr{F}_x \mid$ for some open neighborhood U of x in X, s is induced by some $t \in \Gamma(U, \mathscr{F})$ such that $t_y \in \mathscr{G}_y$ for $y \in U-V\}$. $\mathscr{G}[V]_{\mathscr{F}}$ is coherent and $\operatorname{Supp}(\mathscr{G}[V]_{\mathscr{F}}/\mathscr{G}) \subset V$ (see [10] and [16]).

If $\mathscr{G}$ is an analytic subsheaf of an analytic sheaf $\mathscr{F}$ on a complex analytic space $(X, \mathscr{H})$ and $\mathscr{I}$ is an analytic ideal sheaf on X, then $(\mathscr{G}:\mathscr{I})_{\mathscr{F}}$ denotes the largest analytic subsheaf $\mathscr{S}$ of $\mathscr{F}$ such that $\mathscr{I}\mathscr{S} \subset \mathscr{G}$.

The same notations are also used for the complex algebraic case. When a space has both analytic and algebraic structures, the symbols $\mathcal{O}$ and $\mathscr{H}$ denote respectively the algebraic and analytic structure-sheaves.

All rings are commutative and have identity. Subrings share the same identity elements as the rings they are imbedded in. All modules are unitary. If R is a ring and $N \subset M$ are R-modules and I is an ideal in R, then $(N:I)_M$ is the largest R-submodule S of M such that $IS \subset N$. If $I = Rf$ for some $f \in R$, then $(N:I)_M$ is also denoted by $(N:f)_M$.

LEMMA 1. *Suppose G is an open subset of C^n and V, Y, Z are analytic subvarieties of G. Let $\sigma: G-Y \to G$ be the inclusion map. Suppose $\mathscr{F}$ and $\mathscr{G}$ are coherent analytic sheaves on G such that $O[Y]_{\mathscr{F}} = O[Y]_{\mathscr{G}} = 0$. Suppose $\varphi: \mathscr{F} \to \mathscr{G}$ is a sheaf-homomorphism on G such that φ is an isomorphism on $G-(V \cup Z)$. Suppose p, q are nonnegative integers, f is a holomorphic function on G vanishing identically on V, and Q is a relatively compact open subset of G such that $f^p\sigma_k(\mathscr{G}) = 0$ on $Q-Z$ for $1 \leqq k \leqq q$ and $f^p\sigma_0(\mathscr{G}) \subset \mathscr{G}$ on $Q-Z$. Then there exists $p' \geqq 0$ such that $f^{p'}\sigma_k(\mathscr{F}) = 0$ on $Q-Z$ for $1 \leqq k \leqq q$ and $f^{p'}\sigma_0(\mathscr{F}) \subset \mathscr{F}$ on $Q-Z$.*

Before we prove Lemma 1, we have to clarify one point in its statement. There are canonical maps $\xi: \mathscr{F} \to \sigma_0(\mathscr{F})$ and $\eta: \mathscr{G} \to \sigma_0(\mathscr{G})$. It is obvious that Ker $\xi = O[Y]_{\mathscr{F}}$ and Ker $\eta = O[Y]_{\mathscr{G}}$. Since $O[Y]_{\mathscr{F}} = O[Y]_{\mathscr{G}} = 0$, we can identify $\mathscr{F}$ and $\mathscr{G}$ as subsheaves of $\sigma_0(\mathscr{F})$ and $\sigma_0(\mathscr{G})$ through ξ and η respectively. $f^p\sigma_0(\mathscr{G}) \subset \mathscr{G}$ means that, as subsheaves of $\sigma_0(\mathscr{G})$, $f^p\sigma_0(\mathscr{G})$ is contained in $\mathscr{G}$. $f^{p'}\sigma_0(\mathscr{F}) \subset \mathscr{F}$ has a similar meaning. Similar situations arise frequently in this paper and they should be interpreted similarly.

Proof of Lemma 1. Let $\mathscr{K} = \text{Ker } \varphi$, $\mathscr{R} = \text{Im } \varphi$, and $\mathscr{L} = \text{Coker } \varphi$. Supp $\mathscr{K} \subset V \cup Z$ and Supp $\mathscr{L} \subset V \cup Z$. Let $\mathscr{I} = O[V]_{\mathscr{K}}$ and $\mathscr{J} = O[V]_{\mathscr{L}}$. Supp $\mathscr{I} \subset V$ and Supp $\mathscr{J} \subset V$. Since Q is relatively compact in G and f vanishes identically on V, there exists $r \geqq 0$ such that $f^r\mathscr{I} = f^r\mathscr{J} = 0$ on Q. Since $\mathscr{I} = \mathscr{K}$ on $G-Z$ and $\mathscr{J} = \mathscr{L}$ on $G-Z$, $f^r\mathscr{K} = f^r\mathscr{L} = 0$ on $Q-Z$.

Since $0 \to \mathscr{R} \to \mathscr{G} \to \mathscr{L} \to 0$ is exact, $\sigma_{k-1}(\mathscr{L}) \to \sigma_k(\mathscr{R}) \to \sigma_k(\mathscr{G})$ is exact for $1 \leqq k \leqq q$. Since $f^r\mathscr{L} = 0$ on $Q-Z$, $f^r\sigma_{k-1}(\mathscr{L}) = 0$ on $Q-Z$. Since $f^p\sigma_k(\mathscr{G}) = 0$ on $Q-Z$ for $1 \leqq k \leqq q$, $f^{p+r}\sigma_k(\mathscr{R}) = 0$ on $Q-Z$ for $1 \leqq k \leqq q$.

Since $f^r\mathscr{L} = 0$ on $Q-Z$ and $f^p\sigma_0(\mathscr{G}) \subset \mathscr{G}$ on $Q-Z$, from the exact sequence $0 \to \mathscr{R} \to \mathscr{G} \to \mathscr{L}$ we conclude readily that $f^{p+r}\sigma_0(\mathscr{R}) \subset \mathscr{R}$ on $Q-Z$.

Since $0 \to \mathscr{K} \to \mathscr{F} \to \mathscr{R} \to 0$ is exact, $\sigma_k(\mathscr{K}) \to \sigma_k(\mathscr{F}) \to \sigma_k(\mathscr{R})$ is exact for $1 \leqq k \leqq q$. Since $f^r\mathscr{K} = 0$ on $Q-Z$, $f^r\sigma_k(\mathscr{K}) = 0$ on $Q-Z$. Since $f^{p+r}\sigma_k(\mathscr{R}) = 0$ on $Q-Z$ for $1 \leqq k \leqq q$, $f^{p+2r}\sigma_k(\mathscr{F}) = 0$ on $Q-Z$ for $1 \leqq k \leqq q$.

Since $f^r\sigma_0(\mathscr{K}) = 0$ on $Q-Z$ and $f^{p+r}\sigma_0(\mathscr{R}) \subset \mathscr{R}$ on $Q-Z$, from the exact sequence $\mathscr{K} \to \mathscr{F} \to \mathscr{R} \to 0$ we conclude readily that $f^{p+2r}\sigma_0(\mathscr{F}) \subset \mathscr{F}$ on $Q-Z$.

Hence $p' = p + 2r$ satisfies the requirement. Q.E.D.

LEMMA 2. *Suppose G is an open subset of C^n and Y, V, Z are analytic subvarieties of G such that* $\dim (Y-Z) = r \leqq n-2$ *and $Y \subset V$. Let $\sigma: G-Y \to G$ be the inclusion map. Suppose $\mathscr{F}$ is a coherent analytic sheaf on G such that $O[Y]_{\mathscr{F}} = 0$ and $\mathscr{F}$ is locally free on $G-(V \cup Z)$. If f is a holomorphic function on G vanishing identically on V and Q is a relatively compact open subset of G, then there exists $s \geqq 0$ such that $f^s\sigma_k(\mathscr{F}) = 0$ on $Q-Z$ for $1 \leqq k \leqq n-r-2$ and $f^s\sigma_0(\mathscr{F}) \subset \mathscr{F}$ on $Q-Z$.*

Proof. By replacing Y by $(Y-Z)^-$, we can assume w.l.o.g. that $\dim Y = r$.

Let $\mathscr{F}^* = \text{Hom}_{\mathscr{H}}(\text{Hom}_{\mathscr{H}}(\mathscr{F}, \mathscr{H}), \mathscr{H})$, where $\mathscr{H}$ = the analytic structure-sheaf of C^n. Since Q is relatively compact in G, the problem is local in nature. We can assume w.l.o.g. that we have the following exact sequence on G:

$$\mathscr{H}^{p_{n-r-1}} \to \mathscr{H}^{p_{n-r-2}} \to \cdots \to \mathscr{H}^{p_1} \to \mathscr{H}^{p_0} \to \mathrm{Hom}_{\mathscr{H}}(\mathscr{F}, \mathscr{H}) \to 0.$$

By applying the functor $\mathrm{Hom}_{\mathscr{H}}(\cdot, \mathscr{H})$, we obtain

$$(1) \qquad 0 \longrightarrow \mathscr{F}^* \longrightarrow \mathscr{H}^{p_0} \xrightarrow{\varphi_0} \mathscr{H}^{p_1} \xrightarrow{\varphi_1} \cdots \xrightarrow{\varphi_{n-r-3}} \mathscr{H}^{p_{n-r-2}} \xrightarrow{\varphi_{n-r-2}} \mathscr{H}^{p_{n-r-1}}.$$

Since $\mathscr{F}$ is locally free on $G-(V \cup Z)$, $\mathrm{Hom}_{\mathscr{H}}(\mathscr{F}, \mathscr{H})$ is locally free on $G-(V \cup Z)$. (1) is therefore exact on $G-(V \cup Z)$.

Let $\mathscr{G}_\nu = \mathrm{Ker}\,\varphi_\nu$ and $\mathscr{L}_\nu = \mathrm{Im}\,\varphi_{\nu-1}$. Since $r \leqq n-2$, $\sigma_0(\mathscr{H}) = \mathscr{H}$. Hence, for $\nu \geqq 0$, $\sigma_0(\mathscr{G}_\nu) = \mathscr{G}_\nu[Y]_{\mathscr{H}^{p_\nu}}$ and $\sigma_0(\mathscr{L}_{\nu+1}) = \mathscr{L}_{\nu+1}[Y]_{\mathscr{H}^{p_{\nu+1}}}$. Since φ_ν is defined by a matrix of holomorphic functions, $\mathscr{G}_\nu[Y]_{\mathscr{H}^{p_\nu}} = \mathscr{G}_\nu$. Therefore

$$(2) \qquad \sigma_0(\mathscr{G}_\nu) = \mathscr{G}_\nu \quad \text{for } \nu \geqq 0.$$

Since $\mathrm{Supp}\,(\mathscr{L}_{\nu+1}[Y]_{\mathscr{H}^{p_{\nu+1}}}/\mathscr{L}_{\nu+1}) \subset Y$, there exists a nonnegative integer α_ν such that

$$(3) \qquad f^{\alpha_\nu}\sigma_0(\mathscr{L}_{\nu+1}) \subset \mathscr{L}_{\nu+1} \quad \text{on } Q.$$

For $0 \leqq \nu \leqq n-r-2$ consider:

$$(4)_\nu \qquad \begin{array}{l} \text{for some nonnegative } s_\nu, f^{s_\nu}\sigma_k(\mathscr{G}_\nu) = 0 \\ \text{on } Q-Z \text{ for } 1 \leqq k \leqq n-r-2-\nu. \end{array}$$

We are going to prove $(4)_\nu$ by descending induction on ν. $(4)_{n-r-2}$ is a vacuous statement and hence is true.

For the general case, suppose $0 \leqq \nu < n-r-2$ and $(4)_{\nu+1}$ is true. Since (1) is exact on $G-(V \cup Z)$, $\mathscr{L}_{\nu+1}$ agrees with $\mathscr{G}_{\nu+1}$ on $G-(V \cup Z)$. By (2) and Lemma 1, there exists a nonnegative integer t_ν such that $f^{t_\nu}\sigma_k(\mathscr{L}_{\nu+1}) = 0$ on $Q-Z$ for $1 \leqq k \leqq n-r-2-(\nu+1)$.

$0 \to \mathscr{G}_\nu \to \mathscr{H}^{p_\nu} \to \mathscr{L}_{\nu+1} \to 0$ is exact. For $1 < k \leqq n-r-2-\nu$, $\sigma_{k-1}(\mathscr{L}_{\nu+1}) \to \sigma_k(\mathscr{G}_\nu) \to \sigma_k(\mathscr{H}^{p_\nu}) = 0$ is exact (Korollar zu Satz 3, p. 351, [5]). Hence $f^{t_\nu}\sigma_k(\mathscr{G}_\nu) = 0$ on $Q-Z$ for $1 < k \leqq n-r-2-\nu$. $\sigma_0(\mathscr{H}^{p_\nu}) \to \sigma_0(\mathscr{L}_{\nu+1}) \to \sigma_1(\mathscr{G}_\nu) \to \sigma_1(\mathscr{H}^{p_\nu}) = 0$ is exact (Korollar zu Satz 3, p. 351, [5]). Since $\sigma_0(\mathscr{H}^{p_\nu}) = \mathscr{H}^{p_\nu}$, $\mathrm{Im}\,(\sigma_0(\mathscr{H}^{p_\nu}) \to \sigma_0(\mathscr{L}_{\nu+1})) = \mathscr{L}_{\nu+1}$. By (3), $f^{\alpha_\nu}\sigma_1(\mathscr{G}_\nu) = 0$ on Q. Hence $(4)_\nu$ is satisfied with $s_\nu \geqq \max(\alpha_\nu, t_\nu)$. The induction process is complete.

Since (1) is exact on $G-(V \cup Z)$, $\mathscr{F}^*$ agrees with $\mathscr{G}_0$ on $G-(V \cup Z)$. By (2), $(4)_0$, and Lemma 1, for some nonnegative integer s', $f^{s'}\sigma_0(\mathscr{F}^*) \subset \mathscr{F}^*$ on $Q-Z$ and $f^{s'}\sigma_k(\mathscr{F}^*) = 0$ on $Q-Z$ for $1 \leqq k \leqq n-r-2$.

Since $\mathscr{F}$ is locally free on $G-(V \cup Z)$, the natural map $\mathscr{F} \to \mathscr{F}^*$ is isomorphic on $G-(V \cup Z)$. By Lemma 1, there exists a nonnegative integer s such that $f^s\sigma_0(\mathscr{F}) \subset \mathscr{F}$ on $Q-Z$ and $f^s\sigma_k(\mathscr{F}) = 0$ on $Q-Z$ for $1 \leqq k \leqq n-r-2$. Q.E.D.

PROPOSITION 1. *Suppose G is an open subset of* $\mathbf{C}^n$ *and* Y, V, Z *are analytic subvarieties of G such that* $Y \subset V$. *Let* $\sigma\colon G-Y \to G$ *be the inclusion map. Suppose* l *is a nonnegative integer and* $\mathscr{F}$ *is a coherent analytic sheaf on G such that* $O[Y]_{\mathscr{F}}=0$ *and* $\operatorname{codh}\mathscr{F} \geqq \dim(Y-Z)+l+2$ *on* $G-(V\cup Z)$. *If f is a holomorphic function on G vanishing identically on V and Q is a relatively compact open subset of G, then there exists a nonnegative integer s such that* $f^s\sigma_k(\mathscr{F})=0$ *on* $Q-Z$ *for* $1\leqq k\leqq l$ *and* $f^s\sigma_0(\mathscr{F})\subset\mathscr{F}$ *on* $Q-Z$.

Proof. We can assume w.l.o.g. that $\mathscr{F}\neq 0$. Let $r=\dim(Y-Z)$. Then $l\leqq n-r-2$. We are going to prove the proposition by descending induction on l.

When $l=n-r-2$, $\mathscr{F}$ is locally free on $G-(V\cup Z)$ and the proposition follows from Lemma 2.

For the general case, suppose $l<n-r-2$. Since Q is relatively compact, the problem is local in nature. We can assume w.l.o.g. that we have an exact sequence $0\to\mathscr{G}\to\mathscr{H}^p\to\mathscr{F}\to 0$ on G, where $\mathscr{H}$ is the analytic structure-sheaf of $\mathbf{C}^n$. $\operatorname{codh}\mathscr{G}\geqq r+l+3=r+(l+1)+2$ on $G-(V\cup Z)$.

By induction hypothesis, for some nonnegative integer s, $f^s\sigma_k(\mathscr{G})=0$ on $Q-Z$ for $1\leqq k\leqq l+1$ and $f^s\sigma_0(\mathscr{G})\subset\mathscr{G}$ on $Q-Z$. For $1\leqq k\leqq l$, $0=\sigma_k(\mathscr{H}^p)\to\sigma_k(\mathscr{F})\to\sigma_{k+1}(\mathscr{G})$ is exact. Hence $f^s\sigma_k(\mathscr{F})=0$ on $Q-Z$ for $1\leqq k\leqq l$. $\mathscr{H}^p=\sigma_0(\mathscr{H}^p)\to\sigma_0(\mathscr{F})\to\sigma_1(\mathscr{G})$ is exact. $f^s\sigma_0(\mathscr{F})\subset\mathscr{F}$ on $Q-Z$. Q.E.D.

PROPOSITION 2. *Suppose V is an analytic subvariety of a complex analytic space X and* $\mathscr{F}$ *is a coherent analytic sheaf on X such that* $O[V]_{\mathscr{F}}=0$. *Let* $\theta\colon X-V\to X$ *be the inclusion map. Suppose q is a nonnegative integer and f is a holomorphic function on X such that* $f\theta_0(\mathscr{F})\subset\mathscr{F}$ *and* $f\theta_k(\mathscr{F})=0$ *for* $1\leqq k\leqq q$. *Then* $f^{q+1}H^q(X-V,\mathscr{F})\subset\operatorname{Im}(H^q(X,\mathscr{F})\to H^q(X-V,\mathscr{F}))$.

Proof. Since $f\theta_0(\mathscr{F})\subset\mathscr{F}$, the case $q=0$ is trivial. Hence we assume w.l.o.g. that $q\geqq 1$.

Let

$$0\longrightarrow\mathscr{F}\longrightarrow\mathscr{S}_0\xrightarrow{\varphi_0}\mathscr{S}_1\xrightarrow{\varphi_1}\mathscr{S}_2\xrightarrow{\varphi_2}\cdots$$

be a flabby sheaf resolution on $\mathscr{F}$ on X. Applying the functor $\theta_0(\cdot)$, we obtain

$$0\longrightarrow\mathscr{F}^*\longrightarrow\mathscr{S}_0^*\xrightarrow{\varphi_0^*}\mathscr{S}_1^*\xrightarrow{\varphi_1^*}\mathscr{S}_2^*\xrightarrow{\varphi_2^*}\cdots.$$

This new sequence is in general not exact. Let $\mathscr{K}_\nu=\operatorname{Ker}\varphi_\nu^*$ for $\nu\geqq 0$ and $\mathscr{R}_\nu=\operatorname{Im}\varphi_{\nu-1}^*$ for $\nu\geqq 1$. Then $\mathscr{F}^*\approx\operatorname{Ker}\varphi_0^*$ and $\theta_\nu(\mathscr{F})\approx\mathscr{K}_\nu/\mathscr{R}_\nu$ for $\nu\geqq 1$.

Let $\mathscr{G}_\nu=\operatorname{Ker}\varphi_\nu$. The natural sheaf-homomorphism $\mathscr{S}_\nu\to\mathscr{S}_\nu^*$ induces natural sheaf-homomorphisms $\mathscr{G}_\nu\to\mathscr{K}_\nu$ $(\nu\geqq 0)$ and $\mathscr{G}_\nu\to\mathscr{R}_\nu$ $(\nu\geqq 1)$.

The following commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathscr{G}_\nu & \longrightarrow & \mathscr{S}_\nu & \longrightarrow & \mathscr{G}_{\nu+1} & \longrightarrow & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{K}_\nu & \longrightarrow & \mathscr{S}_\nu^* & \longrightarrow & \mathscr{R}_{\nu+1} & \longrightarrow & 0
\end{array}$$

gives rise to the following commutative diagram with exact rows

$$\begin{array}{ccccccc} H^{q-\nu-1}(X, \mathscr{S}_\nu) & \longrightarrow & H^{q-\nu-1}(X, \mathscr{G}_{\nu+1}) & \longrightarrow & H^{q-\nu}(X, \mathscr{G}_\nu) & \longrightarrow & H^{q-\nu}(X, \mathscr{S}_\nu) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^{q-\nu-1}(X, \mathscr{S}_\nu^*) & \longrightarrow & H^{q-\nu-1}(X, \mathscr{R}_{\nu+1}) & \longrightarrow & H^{q-\nu}(X, \mathscr{K}_\nu) & \longrightarrow & H^{q-\nu}(X, \mathscr{S}_\nu^*). \end{array}$$

Since $\mathscr{S}_\nu$ and $\mathscr{S}_\nu^*$ are both flabby, $H^{q-\nu-1}(X, \mathscr{S}_\nu)=H^{q-\nu}(X, \mathscr{S}_\nu)=H^{q-\nu-1}(X, \mathscr{S}_\nu^*)$ $=H^{q-\nu}(X, \mathscr{S}_\nu^*)=0$ for $0\leqq\nu<q-1$. Hence

$$\begin{array}{ccc} H^{q-\nu-1}(X, \mathscr{G}_{\nu+1}) & \approx & H^{q-\nu}(X, \mathscr{G}_\nu) \\ \downarrow & & \downarrow \\ H^{q-\nu-1}(X, \mathscr{R}_{\nu+1}) & \approx & H^{q-\nu}(X, \mathscr{K}_\nu) \end{array} \tag{5}$$

is a commutative diagram for $0\leqq\nu<q-1$.

We are going to prove $(6)_\nu$ for $0\leqq\nu\leqq q-1$ by induction on ν:

$(6)_\nu$ $$f^{\nu+1}H^{q-\nu}(X, \mathscr{K}_\nu) \subset \operatorname{Im}(H^{q-\nu}(X, \mathscr{G}_\nu) \to H^{q-\nu}(X, \mathscr{K}_\nu)).$$

The exact sequence $0 \to \mathscr{F} \to \mathscr{F}^* \to \mathscr{F}^*/\mathscr{F} \to 0$ gives rise to the exact sequence $H^q(X, \mathscr{F}) \to H^q(X, \mathscr{F}^*) \to H^q(X, \mathscr{F}^*/\mathscr{F})$. Since $f(\mathscr{F}^*/\mathscr{F})=0, fH^q(X, \mathscr{F}^*/\mathscr{F})=0$. $fH^q(X, \mathscr{F}^*)\subset\operatorname{Im}(H^q(X, \mathscr{F}) \to H^q(X, \mathscr{F}^*))$. $(6)_0$ follows from $\mathscr{G}_0\approx\mathscr{F}$ and $\mathscr{K}_0\approx\mathscr{F}^*$.

For the general case, assume $0<\nu\leqq q-1$ and that $(6)_{\nu-1}$ is true. By (5) and $(6)_{\nu-1}$,

$$f^\nu H^{q-\nu}(X, \mathscr{R}_\nu) \subset \operatorname{Im}(H^{q-\nu}(X, \mathscr{G}_\nu) \to H^{q-\nu}(X, \mathscr{R}_\nu)). \tag{7}$$

The followimg commutative diagram with exact rows

$$\begin{array}{ccccccccc} & & \mathscr{G}_\nu & = & \mathscr{G}_\nu & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \longrightarrow & \mathscr{R}_\nu & \longrightarrow & \mathscr{K}_\nu & \longrightarrow & \theta_\nu(\mathscr{F}) & \longrightarrow & 0 \end{array}$$

gives rise to the following commutative diagram with exact rows

$$\begin{array}{ccccc} H^{q-\nu}(X, \mathscr{G}_\nu) & = & H^{q-\nu}(X, \mathscr{G}_\nu) & & \\ \downarrow & & \downarrow & & \\ H^{q-\nu}(X, \mathscr{R}_\nu) & \longrightarrow & H^{q-\nu}(X, \mathscr{K}_\nu) & \longrightarrow & H^{q-\nu}(X, \theta_\nu(\mathscr{F})). \end{array} \tag{8}$$

Since $f\theta_\nu(\mathscr{F})=0$, $fH^{q-\nu}(X, \theta_\nu(\mathscr{F}))=0$. Hence $(6)_\nu$ follows from (7) and (8). The induction process is completed.

The exact sequence $0\to\mathscr{R}_q\to\mathscr{K}_q\to\theta_q(\mathscr{F})\to 0$ yields the exact sequence $0\to\Gamma(X, \mathscr{R}_q)\to\Gamma(X, \mathscr{K}_q)\to\Gamma(X, \theta_q(\mathscr{F}))$. We identify $\Gamma(X, \mathscr{R}_q)$ as a subset of $\Gamma(X, \mathscr{K}_q)$. Since $f\theta_q(\mathscr{F})=0, f\Gamma(X, \theta_q(\mathscr{F})) = 0$.

$$f\Gamma(X, \mathscr{K}_q) \subset \Gamma(X, \mathscr{R}_q). \tag{9}$$

The following commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathscr{G}_{q-1} & \longrightarrow & \mathscr{S}_{q-1} & \longrightarrow & \mathscr{G}_q & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathscr{K}_{q-1} & \longrightarrow & \mathscr{S}^*_{q-1} & \longrightarrow & \mathscr{R}_q & \longrightarrow & 0 \end{array}$$

yields the following commutative diagram with exact rows

$$\begin{array}{ccccccc} \Gamma(X, \mathscr{S}_{q-1}) & \longrightarrow & \Gamma(X, \mathscr{G}_q) & \longrightarrow & H^1(X, \mathscr{G}_{q-1}) & \longrightarrow & H^1(X, \mathscr{S}_{q-1}) = 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \Gamma(X, \mathscr{S}^*_{q-1}) & \longrightarrow & \Gamma(X, \mathscr{R}_q) & \longrightarrow & H^1(X, \mathscr{K}_{q-1}) & \longrightarrow & H^1(X, \mathscr{S}^*_{q-1}) = 0. \end{array}$$

Since $\operatorname{Im}(\Gamma(X, \mathscr{S}_{q-1}) \to \Gamma(X, \mathscr{G}_q)) = \operatorname{Im}(\Gamma(X, \mathscr{S}_{q-1}) \to \Gamma(X, \mathscr{S}_q))$ and

$$\operatorname{Im}(\Gamma(X, \mathscr{S}^*_{q-1}) \to \Gamma(X, \mathscr{R}_q)) = \operatorname{Im}(\Gamma(X, \mathscr{S}^*_{q-1}) \to \Gamma(X, \mathscr{S}^*_q)),$$

the following diagram is commutative:

$$\begin{array}{ccc} \Gamma(X, \mathscr{G}_q)/\operatorname{Im}(\Gamma(X, \mathscr{S}_{q-1}) \longrightarrow \Gamma(X, \mathscr{S}_q)) & \approx & H_1(X, \mathscr{G}_{q-1}) \\ \downarrow & & \downarrow \\ \Gamma(X, \mathscr{R}_q)/\operatorname{Im}(\Gamma(X, \mathscr{S}^*_{q-1}) \longrightarrow \Gamma(X, \mathscr{S}^*_q)) & \approx & H^1(X, \mathscr{K}_{q-1}) \\ \uparrow\!\!\!\!\downarrow & & \\ \Gamma(X, \mathscr{K}_q)/\operatorname{Im}(\Gamma(X, \mathscr{S}^*_{q-1}) \longrightarrow \Gamma(X, \mathscr{S}^*_q)). & & \end{array} \tag{10}$$

On the other hand, we have

$$\begin{aligned} H^q(X, \mathscr{F}) &\approx \Gamma(X, \mathscr{G}_q)/\operatorname{Im}(\Gamma(X, \mathscr{S}_{q-1}) \to \Gamma(X, \mathscr{S}_q)); \\ H^q(X-V, \mathscr{F}) &\approx \Gamma(X, \mathscr{K}_q)/\operatorname{Im}(\Gamma(X, \mathscr{S}^*_{q-1}) \to \Gamma(X, \mathscr{S}^*_q)), \end{aligned} \tag{11}$$

because

$$\begin{aligned} H^q(X, \mathscr{F}) &\approx \operatorname{Ker}(\Gamma(X, \mathscr{S}_q) \to \Gamma(X, \mathscr{S}_{q+1}))/\operatorname{Im}(\Gamma(X, \mathscr{S}_{q-1}) \to \Gamma(X, \mathscr{S}_q)) \\ &= \Gamma(X, \mathscr{G}_q)/\operatorname{Im}(\Gamma(X, \mathscr{S}_{q-1}) \to \Gamma(X, \mathscr{S}_q)) \end{aligned}$$

and

$$\begin{aligned} &H^q(X-V, \mathscr{F}) \\ &\approx \operatorname{Ker}(\Gamma(X-V, \mathscr{S}_q) \to \Gamma(X-V, \mathscr{S}_{q+1}))/\operatorname{Im}(\Gamma(X-V, \mathscr{S}_{q-1}) \to \Gamma(X-V, \mathscr{S}_q)) \\ &= \operatorname{Ker}(\Gamma(X, \mathscr{S}^*_q) \to \Gamma(X, \mathscr{S}^*_{q+1}))/\operatorname{Im}(\Gamma(X, \mathscr{S}^*_{q-1}) \to \Gamma(X, \mathscr{S}^*_q)) \\ &= \Gamma(X, \mathscr{K}_q)/\operatorname{Im}(\Gamma(X, \mathscr{S}^*_{q-1}) \to \Gamma(X, \mathscr{S}^*_q)). \end{aligned}$$

The proposition follows from $(6)_{q-1}$, (9), (10), and (11). Q.E.D.

PROPOSITION 3. *Suppose q is a nonnegative integer, G is an open subset of C^n, and Q is a relatively compact open subset of G. Suppose V, Z are analytic subvarieties of G and f is a holomorphic function on G vanishing identically on V. Let $\theta\colon G-V \to G$*

be the inclusion map. Suppose $\mathscr{F}$ is a coherent analytic sheaf on G such that (i) $O[V]_{\mathscr{F}}=0$ *and* (ii) $\dim A_k \cap V<k$ *for* $k \geqq 0$, *where* $A_k=\bar{S}_{k+q+1}(\mathscr{F}|G-V)-Z$. *Then there exists a nonnegative integer s such that* $f^s\theta_k(\mathscr{F})=0$ *on* $Q-Z$ *for* $1 \leqq k \leqq q$ *and* $f^s\theta_0(\mathscr{F})\subset\mathscr{F}$ *on* $Q-Z$.

Proof. Let r be the largest integer such that $\operatorname{codh}\mathscr{F}>r+q+1$ on $G-(V\cup Z)$. We prove by descending induction on r. When $r \geqq n-q-1$, $\mathscr{F}=0$ on $G-(V\cup Z)$ and the proposition is trivial.

For the general case, assume $r<n-q-1$. Let $W=A_{r+1}^{-}\cup Z$. Then $\operatorname{codh}\mathscr{F}>r+1+q+1$ on $G-(V\cup W)$. By induction hypothesis, there exists a natural number s_1 such that $f^{s_1}\theta_k(\mathscr{F})=0$ on $Q-W$ for $1\leqq k\leqq q$ and $f^{s_1}\theta_0(\mathscr{F})\subset\mathscr{F}$ on $Q-W$.

Let $Y=V\cap W$. Since $\theta_0(\mathscr{F})=\mathscr{F}$ on $G-V$ and $\theta_k(\mathscr{F})=0$ on $G-V$ for $k\geqq 1$, $f^{s_1}\theta_k(\mathscr{F})=0$ on $Q-Y$ for $1\leqq k\leqq q$ and $f^{s_1}\theta_0(\mathscr{F})\subset\mathscr{F}$ on $Q-Y$. Let $s_2=s_1(q+1)$. By Proposition 2, for $0\leqq k\leqq q$ and for any open subset U of Q, $f^{s_2}H^k(U-V,\mathscr{F})\subset \operatorname{Im}(H^k(U-Y,\mathscr{F})\rightarrow H^k(U-V,\mathscr{F}))$. Let $\sigma: G-Y\rightarrow G$ be the inclusion map. Then

$$f^{s_2}\theta_k(\mathscr{F}) \subset \operatorname{Im}(\sigma_k(\mathscr{F})\rightarrow\theta_k(\mathscr{F})) \quad \text{for } 0 \leqq k \leqq q. \tag{12}$$

Since $\dim(Y-Z)\leqq r$ and $\operatorname{codh}\mathscr{F}\geqq q+r+2$ on $G-(V\cup Z)$, by Proposition 1, there exists a nonnegative integer s_3 such that

$$\begin{aligned} f^{s_3}\sigma_0(\mathscr{F}) &\subset \mathscr{F} \quad \text{on } Q-Z; \\ f^{s_3}\sigma_k(\mathscr{F}) &= 0 \quad \text{on } Q-Z \text{ for } 1 \leqq k \leqq q. \end{aligned} \tag{13}$$

Let $s=s_2+s_3$. The proposition follows from (12) and (13). Q.E.D.

LEMMA 3. *Suppose $\mathscr{F}$ is a coherent analytic sheaf on a complex analytic space $(X,\mathscr{H})$. Let A_k be the k-dimensional component of* Supp $O_{[k]\mathscr{F}}$. *If $x\in X$ and f is an element of $\mathscr{H}_x$ not vanishing identically on any nonempty branch-germ of A_k at x for any $k\geqq 0$, then f is not a zero-divisor for $\mathscr{F}_x$.*

Proof. Suppose the contrary. Then we can find an open neighborhood U of x in X, $g\in\Gamma(U,\mathscr{H})$, and $h\in\Gamma(U,\mathscr{F})$ such that $g_x=f$, $h_x\neq 0$, and $gh=0$. Let $Z=\operatorname{Supp} h$ and $k=\dim_x Z$. Then $k\geqq 0$. By shrinking U, we can assume that $\dim Z=k$. $h\in\Gamma(U, O_{[k]\mathscr{F}})$. $Z\subset\operatorname{Supp} O_{[k]\mathscr{F}}$. Since $\dim \operatorname{Supp} O_{[k]\mathscr{F}}\leqq k$, any k-dimensional branch-germ of Z at x is a k-dimensional branch-germ of Supp $O_{[k]\mathscr{F}}$. g_x vanishes identically on a nonempty branch-germ of A_k at x, because $gh=0$. Contradiction. Q.E.D.

LEMMA 4. *Suppose $\mathscr{F}$ is a coherent analytic sheaf on a complex analytic space $(X,\mathscr{H})$. Then, for any $k\geqq 0$, the k-dimensional component of* Supp $O_{[k]\mathscr{F}}$ *equals the k-dimensional component of $S_k(\mathscr{F})$.*

Proof. Let A_k be the k-dimensional component of Supp $O_{[k]\mathscr{F}}$. Let B_k be the k-dimensional component of $S_k(\mathscr{F})$.

Fix $k \geqq 0$. By Satz I, p. 359 of [5], $O_{[k]\mathscr{F}}=0$ on $X-S_k(\mathscr{F})$. Hence $A_k \subset B_k$. To prove $B_k \subset A_k$, assume the contrary. Then $B_k \not\subset \bigcup_{l=0}^{k} A_l$. Take $x \in B_k - \bigcup_{l=0}^{k} A_l$. Choose an open neighborhood U of x in $X-\bigcup_{l=0}^{k} A_l$ and $f \in \Gamma(U, \mathscr{H})$ such that (i) f vanishes identically on $B_k \cap U$ and (ii) for every $l>k$ and every $y \in U$, f_y does not vanish identically on any nonempty branch-germ of A_l at y. This is possible, because $\dim A_l > \dim B_k$ for $l>k$. By Lemma 3, f_y is not a zero-divisor for $\mathscr{F}_y$ for $y \in U$. We have codh $\mathscr{F}_x \geqq 1$. This is a contradiction if $k=0$. If $k>0$, then $\dim S_{k-1}((\mathscr{F}/f\mathscr{F})|U) \leqq k-1$ [6, Satz 5, p. 81]). There exists

$$z \in B_k \cap U - S_{k-1}((\mathscr{F}/f\mathscr{F})|U).$$

codh $(\mathscr{F}/f\mathscr{F})_z > k-1$. codh $\mathscr{F}_z > k$, contradicting $z \in B_k$. Q.E.D.

Lemma 5. *If Y is a nowhere dense $(n-1)$-dimensional analytic subvariety of an n-dimensional complex analytic space X, then there exists a point x of Y and an open neighborhood U of x in X such that $U-Y$ is Stein.*

Proof. Clearly we can assume that X is reduced. Let X' be the union of all branches of X which have dimension $<n$ and let Y' be the union of all branches of Y which have dimension $<n-1$. By replacing X by $X-X'-Y'$ and Y by $Y-X'-Y'$, we can assume that both X and Y are of pure dimension. Let S be the set of all singular points of X.

Case 1. $Y \not\subset S$. Let T be the set of all singular points of Y. By replacing X by $X-S-T$ and Y by $Y-S-T$, we can assume that both X and Y are regular. Take $x \in Y$. There exists a Stein open neighborhood U of x in X such that $U \cap Y$ is the zero-set of a single holomorphic function on U. $U-Y$ is Stein.

Case 2. $Y \subset S$. Take an $(n-1)$-dimensional branch Y_0 of Y. Since $\dim S \leqq n-1$, Y_0 is a branch of S. Let Y_1 be the union of all branches of Y other than Y_0 and let S_1 be the union of all branches of S other than Y_0. By replacing X by $X-Y_1-S_1$ and Y by $Y-Y_1-S_1$, we can assume that $Y=S$. Let $\pi: \tilde{X} \to X$ be the normalization of X. Let $\tilde{Y}=\pi^{-1}(Y)$. Let $\tilde{T}$ be the set of all singular points of $\tilde{Y}$ and let $\tilde{S}$ be the set of all singular points of $\tilde{X}$. $\dim \tilde{T} \leqq n-2$ and $\dim \tilde{S} \leqq n-2$. Take $x \in Y-\pi(\tilde{T} \cup \tilde{S})$. Let $\pi^{-1}(x)=\{x_1, \ldots, x_k\}$. For $1 \leqq i \leqq k$, there exists a Stein open neighborhood U_i of x_i in $\tilde{X}$ such that (i) $\tilde{Y} \cap U_i$ is the zero-set of a single holomorphic function on U_i and (ii) $U_i \cap U_j = \varnothing$ for $i \neq j$. Choose a Stein open neighborhood U of x in X such that $\pi^{-1}(U) \subset \bigcup_{i=1}^{k} U_i$. $\pi^{-1}(U)-\tilde{Y}$ is Stein. Since $U-Y$ is biholomorphic to $\pi^{-1}(U)-\tilde{Y}$, $U-Y$ is Stein. Q.E.D.

Lemma 6. *Suppose Y is a nowhere dense $(n-1)$-dimensional analytic subvariety of an n-dimensional complex analytic space $(X, \mathscr{H})$. Let $\sigma: X-Y \to X$ be the inclusion map. Suppose $\mathscr{F}$ is a coherent analytic sheaf on X such that* Supp $\mathscr{F}=X$. *Then for some $X \in Y$, $\sigma_0(\mathscr{F})_x$ is not finitely generated over $\mathscr{H}_x$.*

Proof. By Lemma 5, we can choose a point x of Y and an open neighborhood U of x in X such that $U-Y$ is Stein. We are going to prove that $\sigma_0(\mathscr{F})_x$ is not finitely generated over $\mathscr{H}_x$. Suppose the contrary.

Choose a sequence $\{x_k\}_{k=1}^{\infty}$ in $U-Y$ whose limit is x. As a set, $\{x_k\}_{k=1}^{\infty}$ is an analytic subvariety of $U-Y$. Let $\mathscr{I}$ be its analytic ideal-sheaf on $U-Y$.

$$\Gamma(U-Y, \mathscr{F}/\mathscr{I}\mathscr{F}) \approx \prod_{k=1}^{\infty} (\mathscr{F}/\mathscr{I}\mathscr{F})_{x_k}.$$

Since $\mathscr{F}_{x_k} \neq \varnothing$, by Krull-Azumaya Lemma [4, (4.1)], $(\mathscr{F}/\mathscr{I}\mathscr{F})_{x_k} \neq 0$. Choose $s_k \in \Gamma(U-Y, \mathscr{F}/\mathscr{I}\mathscr{F})$ such that $(s_k)_{x_l}=0$ for $l>k$ and $(s_k)_{x_k} \neq 0$.

Since $U-Y$ is Stein, the quotient map $\varphi: \Gamma(U-Y, \mathscr{F}) \to \Gamma(U-Y, \mathscr{F}/\mathscr{I}\mathscr{F})$ is surjective. For some $u_k \in \Gamma(U-Y, \mathscr{F})$, $\varphi(u_k)=s_k$. u_k induces an element v_k in $\sigma_0(\mathscr{F})_x$.

Let M_k be the $\mathscr{H}_x$-submodule of $\sigma_0(\mathscr{F})_x$ generated by $v_1, \ldots, v_k$. Since $\sigma_0(\mathscr{F})_x$ is finitely generated over the Noetherian ring $\mathscr{H}_x$, $M_k = M_{k+1}$ for some $k \geqq 1$. Hence, for some neighborhood D of x in U, there exist $f_1, \ldots, f_k \in \Gamma(D, \mathscr{H})$ such that $u_{k+1} = \sum_{i=1}^{k} f_i u_i$ on $D-Y$. $s_{k+1} = \sum_{i=1}^{k} f_i s_i$ on $D-Y$. $(s_{k+1})_{x_{k+1}}=0$. Contradiction. Q.E.D.

Proof of Theorem A. It is obvious that (i) $\Rightarrow$ (ii). To prove (ii) $\Rightarrow$ (iii) and (iii) $\Rightarrow$ (ii), we can assume w.l.o.g. that X is a Stein open subset of $\boldsymbol{C}^n$ and $O[V]_{\mathscr{F}}=0$.

First we prove the following:

(14) for $k \geqq 0$, $V \cap \operatorname{Supp} O_{[k]\mathscr{F}}$ is nowhere dense in $\operatorname{Supp} O_{[k]\mathscr{F}}$.

Suppose the contrary. Then, for some $k \geqq 0$, we can find an open subset U of X such that $U \cap V \supset U \cap \operatorname{Supp} O_{[k]\mathscr{F}} \neq \varnothing$. On U, $O_{[k]\mathscr{F}} \subset O[V]_{\mathscr{F}}=0$, contradicting $U \cap \operatorname{Supp} O_{[k]\mathscr{F}} \neq \varnothing$.

(a) (ii) $\Rightarrow$ (iii). We prove by induction on q. For the case $q=0$, suppose $\dim V \cap \bar{S}_{k+1}(\mathscr{F}|X-V) \geqq k$ for some $k \geqq 0$. Let A denote the $(k+1)$-dimensional component of $S_{k+1}(\mathscr{F}|X-V)$ and B denote the union of all branches of $S_{k+1}(\mathscr{F}|X-V)$ which have dimension $\leqq k$. Since $S_{k+1}(\mathscr{F}|X-V) = A \cup B$ and $\dim V \cap B^- < k$, we have $A \neq \varnothing$ and $\dim V \cap A^- = k$.

Let $Z = \operatorname{Supp} O_{[k+1]\mathscr{F}}$ and $Y = V \cap Z$. By Lemma 4, $A=$the $(k+1)$-dimensional component of $Z-V$. Hence $\dim Y \geqq k$. Since $\dim Z \leqq k+1$ and by (14) Y is nowhere dense in Z, we have $\dim Y=k$ and $\dim Z=k+1$.

Let $\mathscr{I} = (O : O_{[k+1]\mathscr{F}})_{\mathscr{H}}$. Then $(Z, \mathscr{H}/\mathscr{I})$ is a complex analytic space of dimension $k+1$ and $O_{[k+1]\mathscr{F}}$ can be considered naturally as a coherent analytic sheaf on $(Z, \mathscr{H}/\mathscr{I})$.

Let $\sigma: Z-Y \to Z$ be the inclusion map. By Lemma 6 there exists $x \in Y$ such that $\sigma_0(O_{[k+1]\mathscr{F}})_x$ is not finitely generated over $(\mathscr{H}/\mathscr{I})_x$. Since $\sigma_0(O_{[k+1]\mathscr{F}})_x = \theta_0(O_{[k+1]\mathscr{F}})_x \subset \theta_0(\mathscr{F})_x$, $\theta_0(\mathscr{F})_x$ is not finitely generated over $\mathscr{H}_x$. Contradiction. The case $q=0$ is proved.

For the general case, assume $q>0$. Suppose (iii) does not hold. Let k be a nonnegative integer such that $\dim V \cap \bar{S}_{k+q+1}(\mathscr{F}|X-V) \geqq k$. By induction hypothesis, $\dim V \cap \bar{S}_{l+q+1}(\mathscr{F}|X-V) \leqq l$ for every $l \geqq 0$. Hence there exists $x \in V \cap \bar{S}_{k+q+1}(\mathscr{F}|X-V)$ such that $x \notin \bar{S}_{k+q}(\mathscr{F}|X-V)$.

Choose a Stein open neighborhood D of x in X and an irreducible $(k+1)$-

dimensional analytic subvariety W of $\bar{S}_{k+q+1}(\mathscr{F}|X-V) \cap D$ such that (i) $D \cap \bar{S}_{k+q}(\mathscr{F}|X-V)=\varnothing$ and (ii) $\dim V \cap W=k$. Choose a holomorphic function f on D such that (i) f vanishes identically on W and (ii) f does not vanish identically on any $(l+q+1)$-dimensional branch of $\bar{S}_{l+q+1}(\mathscr{F}|X-V) \cap D$ for $l \geqq k$. Since $q \geqq 1$, the existence of such an f follows from dimension arguments.

By Lemmas 3 and 4, f_y is not a zero-divisor for $\mathscr{F}_y$ for $y \in D-V$. On D, let $\varphi: \mathscr{F} \to \mathscr{F}$ be the sheaf-homomorphism defined by multiplication by f. Then $\operatorname{Ker} \varphi \subset V \cap D$. Since $O[V]_{\mathscr{F}}=0$, $\operatorname{Ker} \varphi=0$.

The exact sequence $0 \to \mathscr{F} \xrightarrow{\varphi} \mathscr{F} \to \mathscr{F}/f\mathscr{F} \to 0$ on D yields the exact sequence $\theta_l(\mathscr{F}) \to \theta_l(\mathscr{F}/f\mathscr{F}) \to \theta_{l+1}(\mathscr{F})$ on D for $l \geqq 0$. Hence $\theta_0(\mathscr{F}/f\mathscr{F})_y, \ldots, \theta_{q-1}(\mathscr{F}/f\mathscr{F})_y$ are finitely generated over $\mathscr{H}_y$ for $y \in V \cap D$.

By induction hypothesis, $\dim V \cap \bar{S}_{l+q}((\mathscr{F}/f\mathscr{F})|D-V)<l$ for every $l \geqq 0$. Since f is identically zero on W and $W-V \subset S_{k+q+1}(\mathscr{F}|D-V)$,

$$W-V \subset S_{k+q}((\mathscr{F}|f\mathscr{F})|D-V)$$

and

$$\dim V \cap \bar{S}_{k+q}((\mathscr{F}/f\mathscr{F})|D-V) \geqq \dim V \cap (W-V)^- = \dim V \cap W = k.$$

Contradiction. The general case is proved.

(b) (iii) $\Rightarrow$ (i). We prove by induction on q. Choose a holomorphic function f on X such that (i) f vanishes identically on V and (ii) f does not vanish identically on any branch of Supp $O_{[k]\mathscr{F}}$ for any $k \geqq 0$. The choice of such an f is possible because of (14).

Fix arbitrarily $x \in V$. Choose a relatively compact open neighborhood Q of x in X.

By Proposition 3, there exists a nonnegative integer s such that $f^s\theta_k(\mathscr{F})=0$ on Q for $1 \leqq k \leqq q$ and $f^s\theta_0(\mathscr{F}) \subset \mathscr{F}$ on Q. Let $\varphi: \mathscr{F} \to \mathscr{F}$ and $\tilde{\varphi}: \theta_0(\mathscr{F}) \to \theta_0(\mathscr{F})$ be defined by multiplication by f^s. By Lemma 3, φ and $\tilde{\varphi}$ are injective.

Since $f^s\theta_0(\mathscr{F}) \subset \mathscr{F}$ on Q, $\tilde{\varphi}(\theta_0(\mathscr{F})) \subset \mathscr{F}$. On Q, $\tilde{\varphi}(\theta_0(\mathscr{F}))=\varphi(\mathscr{F})[V]_{\mathscr{F}}$ is coherent. Since $\tilde{\varphi}$ is injective, $\theta_0(\mathscr{F})$ is coherent on Q. Since x is arbitrary, the case $q=0$ is proved.

For the general case, assume $q>0$. $S_{k+q}((\mathscr{F}/f\mathscr{F})|X-V) \subset S_{k+q+1}(\mathscr{F}|X-V)$. Hence $\dim V \cap \bar{S}_{k+q}((\mathscr{F}/f\mathscr{F})|X-V)<k$ for every $k \geqq 0$. By induction hypothesis, $\theta_k(\mathscr{F})$ and $\theta_k(\mathscr{F}/f\mathscr{F})$ are coherent on X for $0 \leqq k \leqq q-1$.

The exact sequence $0 \to \mathscr{F} \xrightarrow{\varphi} \mathscr{F} \to \mathscr{F}/f\mathscr{F} \to 0$ yields the exact sequence $\theta_{q-1}(\mathscr{F}) \to \theta_{q-1}(\mathscr{F}/f\mathscr{F}) \to \theta_q(\mathscr{F}) \xrightarrow{\psi} \theta_q(\mathscr{F})$. Since $f^s\theta_q(\mathscr{F}) = 0$ on Q, $\psi=0$ on Q. On Q, $\theta_q(\mathscr{F})=\operatorname{Coker}(\theta_{q-1}(\mathscr{F}) \to \theta_{q-1}(\mathscr{F}/f\mathscr{F}))$ is coherent. Since x is arbitrary, the induction process is complete. Q.E.D.

The complex algebraic analog of Theorem A is the following:

THEOREM A$'$. *Suppose V is an algebraic subvariety of a complex algebraic space $(X, \mathcal{O})$, q is a nonnegative integer, and $\mathscr{F}$ is a coherent algebraic sheaf on X. Let $\theta: X-V \to X$ be the inclusion map. Then the following three statements are equivalent*:

(i) $\theta_0(\mathscr{F}|X-V), \ldots, \theta_q(\mathscr{F}|X-V)$ *(or equivalently* $\mathscr{H}_V^0(\mathscr{F}), \ldots, \mathscr{H}_V^{q+1}(\mathscr{F})$*) are coherent on X.*

(ii) *For every* $x \in V$, $\theta_0(\mathscr{F}|X-V)_x, \ldots, \theta_q(\mathscr{F}|X-V)_x$ *(or equivalently* $\mathscr{H}_V^0(\mathscr{F})_x, \ldots, \mathscr{H}_V^{q+1}(\mathscr{F})_x$*) are finitely generated over* $\mathcal{O}_x$.

(iii) $\dim V \cap \bar{S}_{k+q+1}(\mathscr{F}|X-V) < k$ *for every* $k \geqq 0$.

The proof of Theorem A′ is trivially analogous to the proof of Theorem A. There is no need to repeat the argument. However, since the algebraic analog of Scheja's theorem (Korollar zu Satz 3, p. 351, [5]) on extending cohomology classes which is vital in the proof of Theorem A cannot be proved in the same way as the analytic case, we present what we need for the algebraic case in the following proposition.

Proposition 4. *Suppose V is an algebraic subvariety of dimension r in* $\boldsymbol{C}^n$ *and* $\mathcal{O}$ *is the algebraic structure-sheaf of* $\boldsymbol{C}^n$. *Let* $\theta: \boldsymbol{C}^n - V \to \boldsymbol{C}^n$ *be the inclusion map. Then* $\theta_k(\mathcal{O}) = 0$ *for* $1 \leqq k \leqq n-2-r$ *and* $\theta_0(\mathcal{O}) = \mathcal{O}$ *if* $r \leqq n-2$.

Proposition 4 can be obtained from some general theorems in the works of Grothendieck ([2], [3]). However, for the sake of completeness, we present a direct proof here.

In Lemmas 7 and 8 and the proof of Proposition 4 below, Ω denotes $\boldsymbol{C}^n$ and $\mathcal{O}$ denotes the algebraic structure-sheaf of $\boldsymbol{C}^n$. R denotes $\Gamma(\Omega, \mathcal{O})$. If $Q \in R$, then Ω_Q denotes the complement of the zero-set of Q in Ω.

Lemma 7. *Suppose V is an algebraic subvariety of Q,* $Q \in R$, *and* $\mathscr{F}$ *is a coherent algebraic sheaf on* Ω_Q. *Suppose p is a nonnegative integer and* $f \in R$ *vanishes identically on V. If* $s \in H_V^p(\Omega_Q, \mathscr{F})$, *then there exists a nonnegative integer m (depending on s), such that* $f^m s = 0$.

Proof. We are going to prove by induction on p. Let V be the set of all common zeros of $Q_1, \ldots, Q_k \in R$.

(a) The case $p=0$. $H_V^0(\Omega_Q, \mathscr{F})$ is the subset of all elements of $\Gamma(\Omega_Q, \mathscr{F})$ having supports in V. Since $s=0$ on $\Omega_Q - \Omega_f$, by Proposition 6, p. 235 of [7], there exists a nonnegative integer m such that $f^m s = 0$ on Ω_Q.

(b) The case $p=1$, $\Gamma(\Omega_Q, \mathscr{F}) \xrightarrow{\alpha} \Gamma(\Omega_Q - V, \mathscr{F}) \xrightarrow{\beta} H_V^1(\Omega_Q, \mathscr{F}) \to H^1(\Omega_Q, \mathscr{F}) = 0$ is exact. $s = \beta(s^*)$ for some $s^* \in \Gamma(\Omega_Q - V, \mathscr{F})$. Since $\Omega_Q - V \subset \Omega_f$, by Lemma 1, p. 247 of [7], there exists a nonnegative integer r and $\tilde{s} \in \Gamma(\Omega_Q, \mathscr{F})$ such that $\tilde{s} = f^r s^*$ on $\Omega_Q \cap \Omega_f$. Since $\tilde{s} - f^r s^* = 0$ on $\Omega_{Q_i} \cap \Omega_Q \cap \Omega_f$, by Proposition 6, p. 235 of [7], there exists a nonnegative integer q_i such that $f^{q_i}(\tilde{s} - f^r s^*) = 0$ on $\Omega_{Q_i} \cap \Omega_Q$. Let $q = \max(q_1, \ldots, q_k)$. Then $f^q \tilde{s} = f^{q+r} s^*$ on $\Omega_Q - V$, $f^{q+r} s^* \in \operatorname{Im} \alpha$, and $f^{q+r} s = 0$.

(c) For the general case, assume $p > 1$. Then $H_V^p(\Omega_Q, \mathscr{F}) \approx H^{p-1}(\Omega_Q - V, \mathscr{F})$. We identify the two. Let $U_i = \Omega_{Q_i} \cap \Omega_Q$ and $\mathfrak{U} = \{U_i\}_{i=1}^k$. $\mathfrak{U}$ covers $\Omega_Q - V$. s is represented by an element of $Z^{p-1}(\mathfrak{U}, \mathscr{F})$ given by $s_{i_0 \cdots i_{p-1}} \in \Gamma(U_{i_0} \cap \cdots \cap U_{i_{p-1}}, \mathscr{F})$, $1 \leqq i_0, \ldots, i_{p-1} \leqq k$. By Lemma 1, p. 247 of [7], there exist a nonnegative integer r and $t_{i_0 \cdots i_{p-1}} \in \Gamma(\Omega_Q, \mathscr{F})$ such that $(Q_{i_0} \cdots Q_{i_{p-1}})^r s_{i_0 \cdots i_{p-1}} = t_{i_0 \cdots i_{p-1}}$ on $U_{i_0} \cap \cdots$

$\cap U_{i_{p-1}}$. By Hilbert Nullstellensatz, there exist a nonnegative integer m and $g_1, \ldots, g_k \in R$ *such that* $f^m = \sum_{i=1}^k g_i Q_i^r$. Define $u_{i_0 \cdots i_{p-2}} \in \Gamma(U_{i_0} \cap \cdots \cap U_{i_{p-2}}, \mathscr{F})$ by $u_{i_0 \cdots i_{p-2}} = \sum_{i=1}^k g_i t_{i i_0 \cdots i_{p-2}} (Q_{i_0} \cdots Q_{i_{p-2}})^{-r}$. $\{u_{i_0 \cdots i_{p-2}}\}$ defines an element u of $C^{p-2}(\mathfrak{U}, \mathscr{F})$. On $U_{i_0} \cap \cdots \cap U_{i_{p-1}}$,

$$\begin{aligned}(\delta u)_{i_0 \cdots i_{p-1}} &= \sum_{\nu=0}^{p-1} (-1)^\nu u_{i_0 \cdots \hat{i}_\nu \cdots i_{p-1}} \\ &= \sum_{\nu=0}^{p-1} (-1)^\nu \sum_{i=1}^k g_i t_{i i_0 \cdots \hat{i}_\nu \cdots i_{p-1}} (Q_{i_0} \cdots \hat{Q}_{i_\nu} \cdots Q_{i_{p-1}})^{-r} \\ &= \sum_{\nu=0}^{p-1} (-1)^\nu \sum_{i=1}^k g_i Q_i^r s_{i i_0 \cdots \hat{i}_\nu \cdots i_{p-1}} \\ &= \sum_{i=1}^k g_i Q_i^r \sum_{\nu=0}^{p-1} (-1)^\nu s_{i i_0 \cdots \hat{i}_\nu \cdots i_{p-1}} \\ &= \sum_{i=1}^k g_i Q_i^r s_{i_0 \cdots i_{p-1}} = f^m s_{i_0 \cdots i_{p-1}}.\end{aligned}$$

Hence $f^{m_s} = 0$. Q.E.D.

LEMMA 8. *Suppose $f_1, \ldots, f_k \in R$ and f_i is not a zero-divisor for $R/\sum_{\nu=1}^{i-1} Rf_\nu$ for $1 \leqq i \leqq k$, where $\sum_{\nu=1}^0 Rf_\nu = 0$. Suppose I is an ideal in R and* $\dim I < n-k$. *Then there exists $f \in I$ such that f is not a zero-divisor for $R/\sum_{\nu=1}^k Rf_\nu$.*

Proof. If $R = \sum_{\nu=1}^k Rf_\nu$, then there is nothing to prove. Assume $R \neq \sum_{\nu=1}^k Rf_\nu$. Since f_i is not a zero-divisor for $R/\sum_{\nu=1}^{i-1} Rf_\nu$ for $1 \leqq i \leqq k$, the height of any isolated prime ideal of $\sum_{\nu=1}^i Rf_\nu$ is greater than the height of some isolated prime ideal of $\sum_{\nu=1}^{i-1} Rf_\nu$ for $1 \leqq i \leqq k$ [22, Corollary 3, p. 214, Vol. I]. By Theorem 30, p. 240, Vol. I of [22], the height of every isolated prime ideal of $\sum_{\nu=1}^k Rf_\nu$ is equal to k. By Theorem 20, p. 193, Vol. II of [22], $\dim \sum_{\nu=1}^k Rf_\nu = n-k$, because R is simply the polynomial ring of n algebraically independent variables over $\boldsymbol{C}$. Let $\{P_\alpha\}_{\alpha \in A}$ be the set of all associated prime ideals of $\sum_{\nu=1}^k Rf_\nu$. By Theorem 26, p. 203, Vol. II of [22], $\dim P_\alpha = n-k$ for all α. Hence $I \not\subset P_\alpha$ for $\alpha \in A$. $I \not\subset \bigcup_{\alpha \in A} P_\alpha$ [22, p. 215, Vol. I]. Any element f of $I - \bigcup_{\alpha \in A} P_\alpha$ satisfies the requirement [22, Corollary 3, p. 214, Vol. I] Q.E.D.

Proof of Proposition 4. First we prove the following:

(15) If $f_1, \ldots, f_k, f \in R$ and f is not a zero-divisor for $R/\sum_{\nu=1}^k Rf_\nu$, then the sheaf-homomorphism $\varphi: \mathscr{G} \to \mathscr{G}$ defined by multiplication by f is injective, where $\mathscr{G} = \mathscr{O}/\sum_{\nu=1}^k \mathscr{O}f_\nu$.

To prove (15), take arbitrarily $Q \in R$. Suppose $s \in \Gamma(\Omega_Q, \mathscr{G})$ and $\varphi(s) = 0$. By Lemma 1, p. 247 of [7], there exist a nonnegative integer r and $s' \in R(\Omega, \mathscr{G})$ such that $s' = Q^r s$ on Ω_Q. $\varphi(s') = 0$ on Ω_Q. By Proposition 6, p. 235 of [7], there exists a nonnegative integer m such that $\varphi(Q^m s') = Q^m \varphi(s') = 0$ on Ω. Since $H^1(\Omega, \sum_{\nu=1}^k \mathscr{O}f_\nu) = 0$ [7, Corollary 1, p. 239], $Q^m s'$ is the image of some $s^* \in R$ under the quotient map $\mathscr{O} \to \mathscr{G}$. Let $\alpha: \mathscr{O}^k \to \sum_{\nu=1}^k \mathscr{O}f_\nu$ be defined by $\alpha(0, \ldots, 0, 1, 0, \ldots, 0) = f_\nu$, where 1 is in the νth place. Since $H^1(\Omega, \operatorname{Ker} \alpha) = 0$, $fs^* = \sum_{\nu=1}^k g_\nu f_\nu$ for some

$g_1, \ldots, g_k \in R$. Since f is not a zero-divisor for $R/\sum_{\nu=1}^k Rf_\nu$, $s^* = \sum_{\nu=1}^k h_\nu f_\nu$ for some $h_1, \ldots, h_k \in R$. $Q^m s' = 0$ on Ω, and $s=0$ on Ω_Q. (15) is proved.

Fix arbitrarily $Q \in R$. Let I be the ideal of R associated to V. For $0 \leqq k \leqq n-r$, consider

$(16)_k$ If $f_1, \ldots, f_k \in I$ and f_i is not a zero-divisor for $R/\sum_{\nu=1}^{i-1} Rf_\nu$ for $1 \leqq i \leqq k$ (where $\sum_{\nu=1}^0 Rf_\nu = 0$), then $H_V^p(\Omega_Q, \mathcal{O}_k) = 0$ for $0 \leqq p < n-k-r$, where $\mathcal{O}_k = \mathcal{O}/\sum_{\nu=1}^k \mathcal{O}f_\nu$ and $\mathcal{O}_0 = \mathcal{O}$.

We are going to prove $(16)_k$ by descending induction on k. $(16)_{n-r}$ is vacuous. To prove the general case, assume $0 \leqq k < n-r$. Suppose there exists a *nonzero* element s in $H_V^p(\Omega_Q, \mathcal{O}_k)$ for some $0 \leqq p < n-k-r$. Since $\dim V = r$, $\dim I = r < n-k$. By Lemma 8, there exists $f \in I$ such that f is not a zero-divisor for $R/\sum_{\nu=1}^k Rf_\nu$. By Lemma 7, there exists a nonnegative integer m such that $f^m s = 0$. Let $f_{k+1} = f^m$. By (15) $0 \to \mathcal{O}_k \xrightarrow{\varphi} \mathcal{O}_k \to \mathcal{O}_{k+1} \to 0$ is exact, where φ is defined by multiplication by f_{k+1} and $\mathcal{O}_{k+1} = \mathcal{O}/\sum_{\nu=1}^{k+1} \mathcal{O}f_\nu$.

$$H_V^{p-1}(\Omega_Q, \mathcal{O}_{k+1}) \longrightarrow H_V^p(\Omega_Q, \mathcal{O}_k) \xrightarrow{\tilde{\varphi}} H_V^p(\Omega_Q, \mathcal{O}_k)$$

is exact, where $H_V^{-1}(\Omega_Q, \mathcal{O}_{k+1}) = 0$. By $(16)_{k+1}$, $H_V^{p-1}(\Omega_Q, \mathcal{O}_{k+1}) = 0$. Hence $\tilde{\varphi}$ is injective. $f^m s = \tilde{\varphi}(s) \neq 0$. Contradiction. $(16)_k$ is true for $0 \leqq k \leqq n-r$.

The proposition follows from $(16)_0$. Q.E.D.

In (α), (β), (γ) and (δ) below, we suppose the following: B is a ring. A is a subring of B. B is A-flat as an A-module (Definition 3, p. 34, [8]). M is an A-module. N and N' are A-submodules of M. Canonically isomorphic modules are identified.

(α) $N \otimes_A B$ can be regarded canonically as a B-submodule of $M \otimes_A B$ and $(M/N) \otimes_A B = (M \otimes_A B)/(N \otimes_A B)$.

(α) follows from tensoring the exact sequence $0 \to N \to M \to M/N \to 0$ with B over A.

(β) $(N \cap N') \otimes_A B = (N \otimes_A B) \cap (N' \otimes_A B)$.

Let $\varphi: M \to M/N$ and $\varphi': M \to M/N'$ be the quotient maps and define $\varphi: M \to (M/N) \oplus (M/N')$ by $\varphi(f) = \varphi(f) \oplus \varphi'(f)$. By tensoring the exact sequence $0 \to N \cap N' \to M \xrightarrow{\varphi} (M/N) \oplus (M/N')$ with B over A, we obtain the exact sequence $0 \to (N \cap N') \otimes_A B \to M \otimes_A B \to [(M/N) \oplus (M/N')] \otimes_A B$. By (α), $[(M/N) \oplus (M/N)] \otimes_A B = [(M \otimes_A B)/(N \otimes_A B)] \oplus [(M \otimes_A B)/(N' \otimes_A B)]$. On the other hand, the kernel of $M \otimes_A B \to [(M \otimes_A B)/(N \otimes_A B)] \oplus [(M \otimes_A B)/(N' \otimes_A B)]$ is $(N \otimes_A B) \cap (N' \otimes_A B)$. (β) follows.

(γ) $[(N:f)_M] \otimes_A B = [(N \otimes_A B):f]_{M \otimes_A B}$.

Consider the exact sequence $0 \to (N:f)_M \to M \xrightarrow{\varphi} M/N$, where φ is the composite of the quotient map $M \to M/N$ and the map $M \to M$ defined by multiplication by f. Tensoring it with B over A yields the exact sequence $0 \to [(N:f)_M] \otimes_A B \to M \otimes_A B \to (M/N) \otimes_A B$. By (α), $(M/N) \otimes_A B = (M \otimes_A B)/(N \otimes_A B)$. On the other hand, the kernel of $M \otimes_A B \to (M \otimes_A B)/(N \otimes_A B)$ is $([N \otimes_A B]:f)_{M \otimes_A B}$. (γ) follows.

(δ) If I is a finitely generated ideal in A, then

$$[(N:I)_M] \otimes_A B = ([N \otimes_A B]:[I \otimes_A B])_{M \otimes_A B}.$$

Suppose $I = \sum_{i=1}^k Af_i$. $(N:I)_M = \bigcap_{i=1}^k (N:f_i)_M$ and $([N \otimes_A B]:[I \otimes_A B])_{M \otimes_A B} = \bigcap_{i=1}^k ([N \otimes_A B]:f_i)_{M \otimes_A B}$. ($\delta$) follows from ($\beta$) and ($\gamma$).

Proposition 5. *Suppose A is a Noetherian subring of a Noetherian ring B and B is A-flat as an A-module. Suppose I is an ideal in A and N is an A-submodule of a finitely generated A-module M. Then*

$$\left[\bigcup_{k=1}^{\infty} (N:I^k)_M\right] \otimes_A B = \bigcup_{k=1}^{\infty} ([N \otimes_A B]:[I \otimes_A B]^k)_{M \otimes_A B}.$$

Proof. $(N:I)_M \subset (N:I^2)_M \subset \cdots \subset (N:I^k)_M \subset \cdots$ and

$$([N \otimes_A B]:[I \otimes_A B])_{M \otimes_A B} \subset ([N \otimes_A B]:[I \otimes_A B]^2)_{M \otimes_A B} \subset \cdots$$
$$\subset ([N \otimes_A B]:[I \otimes_A B]^k)_{M \otimes_A B} \subset \cdots$$

are both eventually stationary, because M is finitely generated over the Noetherian ring A and $M \otimes_A B$ is finitely generated over the Noetherian ring B. For some $l \geqq 1$, $\bigcup_{k=1}^{\infty} (N:I^k)_M = (N:I^l)_M$ and

$$\bigcup_{k=1}^{\infty} ([N \otimes_A B]:[I \otimes_A B]^k)_{M \otimes_A B} = ([N \otimes_A B]:[I \otimes_A B]^l)_{M \otimes_A B}.$$

The lemma follows from (δ). Q.E.D.

In (ε), (ζ) and (η) below, we assume the following: (A, B) is a flat couple of local rings [8, Definition 4, p. 36]. M is an A-module. $\tilde{M} = M \otimes_A B$. Since the natural map $M \to \tilde{M}$ is injective [8, Proposition 22, p. 36], M is identified canonically as a subset of $\tilde{M}$.

(ε) If $f \in A$ is not a zero-divisor for M, then f is not a zero-divisor for $\tilde{M}$.

Since the homomorphism $M \to M$ defined by multiplication by f is injective and B is A-flat, the homomorphism $\tilde{M} \to \tilde{M}$ defined by multiplication by f is injective. (ε) follows.

(ζ) If I is an ideal in A, then $(I\tilde{M}) \cap M = IM$.

(ζ) is the consequence of $(M/IM) \otimes_A B = \tilde{M}/I\tilde{M}$ and the injectivity of the canonical homomorphism $M/IM \to (M/IM) \otimes_A B$ [8, Proposition 22, p. 36].

(η) If $f_1, \ldots, f_k$ are in the intersection of the maximal ideal of A and B and $f_1, \ldots, f_k$ form an M-sequence [4, p. 94], then $f_1, \ldots, f_k$ form an $\tilde{M}$-sequence.

For $1 \leqq i \leqq k$, f_i is not a zero-divisor for $M/\sum_{v=1}^{i-1} f_v M$. By (ε), f_i is not a zero-divisor for $(M/\sum_{v=1}^{i-1} f_v M) \otimes_A B = \tilde{M}/\sum_{v=1}^{i-1} f_v \tilde{M}$ for $1 \leqq i \leqq k$. (η) follows.

Proposition 6. *Suppose $(A, \mathfrak{m})$ and $(B, \mathfrak{n})$ are local rings such that (A, B) is a flat couple and $\mathfrak{m}B = \mathfrak{n}$. If M is a finitely generated A-module and $\tilde{M} = M \otimes_A B$, then* $\operatorname{codh}_A \tilde{M} = \operatorname{codh}_B \tilde{M}$.

Proof. Let $\operatorname{codh}_A M = k$. Let $f_1, \ldots, f_k$ be an M-sequence. (In the case $k = 0$, $f_1, \ldots, f_k$ is just the empty sequence and the argument that follows applies also to this case.) By (η), $f_1, \ldots, f_k$ is an $\tilde{M}$-sequence. We need only prove that $f_1, \ldots, f_k$ is a maximal $\tilde{M}$-sequence.

Since $f_1, \ldots, f_k$ is a maximal M-sequence, every element of $\mathfrak{m}$ is a zero-divisor for $M/\sum_{v=1}^k f_v M$. $\mathfrak{m}$ is an associated prime ideal of the A-submodule $\sum_{v=1}^k f_v M$ of the A-module M. There exist $g \in M$ and a natural number r such that $g \notin \sum_{v=1}^k f_v M$ and $\mathfrak{m}^r g \subset \sum_{v=1}^k f_v M$. By ($\zeta$), $g \notin \sum_{v=1}^k f_v \tilde{M}$. Since $\mathfrak{n} = \mathfrak{m}B$, $\mathfrak{n}^r g \subset \sum_{v=1}^k f_v \tilde{M}$. Hence $\mathfrak{n}$ is an associated prime ideal of the B-submodule $\sum_{v=1}^k f_v \tilde{M}$ of the B-module $\tilde{M}$. Every element of $\mathfrak{n}$ is a zero-divisor for $\tilde{M}/\sum_{v=1}^k f_v \tilde{M}$. $f_1, \ldots, f_k$ is a maximal $\tilde{M}$-sequence. Q.E.D.

PROPOSITION 7. *Suppose X is a complex algebraic space, V is an algebraic subvariety of X, and $\mathcal{F}$ is a coherent algebraic sheaf on X. Then the canonical homomorphism $\mathcal{H}_V^0(\mathcal{F})^h \to \mathcal{H}_V^0(\mathcal{F}^h)$ is an isomorphism.*

Proof. It is clear from the definitions that $\mathcal{H}_V^0(\mathcal{F}) = O[V]_{\mathcal{F}}$ and $\mathcal{H}_V^0(\mathcal{F}^h) = O[V]_{\mathcal{F}^h}$.

Let $\mathcal{O}$ be the algebraic structure-sheaf of X and $\mathcal{H}$ be the associated analytic structure-sheaf. Let $\mathcal{I}$ be the algebraic ideal-sheaf of V. Then $\mathcal{I}^h$ is the analytic ideal-sheaf of V [8, Proposition 4, p. 10]. By [10, Theorem 1, p. 376], $\mathcal{O}[V]_{\mathcal{F}^h} = \bigcup_{k=1}^{\infty} (O:(\mathcal{I}^h)^k)_{\mathcal{F}^h}$. Analogously it can be proved that $O[V]_{\mathcal{F}} = \bigcup_{k=1}^{\infty} (O:\mathcal{I}^k)_{\mathcal{F}}$.

For every $x \in X$, $(\mathcal{O}_x, \mathcal{H}_x)$ is a flat couple [8, Corollary 1, p. 11]. The proposition follows from Proposition 5. Q.E.D.

Proof of Theorem B. Let $\mathcal{O}$ be the algebraic structure-sheaf of X and let $\mathcal{H}$ be the associated analytic structure-sheaf. Since $(\mathcal{O}_x, \mathcal{H}_x)$ is a flat couple for every $x \in X$, $S_k(\mathcal{F}|X-V) = S_k(\mathcal{F}^h|X-V)$ for every $k \geqq 0$ by Proposition 6. Hence the equivalence of the coherence of $\mathcal{H}_V^0(\mathcal{F}), \ldots, \mathcal{H}_V^{q+1}(\mathcal{F})$ and the coherence of $\mathcal{H}_V^0(\mathcal{F}^h), \ldots, \mathcal{H}_V^{q+1}(\mathcal{F}^h)$ follows from Theorems A and A′.

Suppose we have the coherence of either $\mathcal{H}_V^0(\mathcal{F}), \ldots, \mathcal{H}_V^{q+1}(\mathcal{F})$ or $\mathcal{H}_V^0(\mathcal{F}^h), \ldots, \mathcal{H}_V^{q+1}(\mathcal{F}^h)$ (and hence the coherence of all). We are going to prove the bijectivity of $\mathcal{H}_V^k(\mathcal{F})^h \to \mathcal{H}_V^k(\mathcal{F}^h)$ for $0 \leqq k \leqq q+1$ by induction on k. The case $k=0$ follows from Proposition 7. For the general case fix $0 < k \leqq q+1$.

Let $\mathcal{K} = O[V]_{\mathcal{F}}$ and $\theta: X-V \to X$ be the inclusion map. Since $\operatorname{Supp} \mathcal{K} \subset V$, $\theta_l(\mathcal{K}) = 0$ for $l \geqq 0$. Hence $H_V^1(\mathcal{K}) \approx \operatorname{Coker}(\mathcal{K} \to \theta_0(\mathcal{K})) = 0$ and $H_V^l(\mathcal{K}) \approx \theta_{l-1}(\mathcal{K}) = 0$ for $l > 1$. The exact sequence $\mathcal{H}_V^l(\mathcal{K}) \to \mathcal{H}_V^l(\mathcal{F}) \to \mathcal{H}_V^l(\mathcal{F}/\mathcal{K}) \to \mathcal{H}_V^{l+1}(\mathcal{K})$ implies that $\mathcal{H}_V^l(\mathcal{F}) \approx \mathcal{H}_V^l(\mathcal{F}/\mathcal{K})$ for $l \geqq 1$. Likewise we have $\mathcal{H}_V^l(\mathcal{F}^h) \approx \mathcal{H}_V^l((\mathcal{F}/\mathcal{K})^h)$ for $l \geqq 1$. Hence, by replacing $\mathcal{F}$ by $\mathcal{F}/\mathcal{K}$, we can assume that $O[V]_{\mathcal{F}} = 0$.

We can assume that X is an algebraic subspace of some $\boldsymbol{C}^N$. Choose $f \in \Gamma(X, \mathcal{O})$ such that (i) f vanishes identically on V and (ii) for any $l \geqq 0$, f does not vanish identically on any branch of $\operatorname{Supp} O_{[l]\mathcal{F}}$ not contained in V. Let $\varphi: \mathcal{F} \to \mathcal{F}$ be defined by multiplication by f. $\operatorname{Ker} \varphi = 0$ on $X-V$ (algebraic analog of Lemma 3). Since $O[V]_{\mathcal{F}} = 0$, $\operatorname{Ker} \varphi = 0$ on X.

Let Q be an arbitrary relatively compact open subset of X. Since $\mathcal{H}_V^k(\mathcal{F})$ and $\mathcal{H}_V^k(\mathcal{F}^h)$ are coherent and f vanishes identically on their supports, there exists a natural number m such that $f^m \mathcal{H}_V^k(\mathcal{F}) = 0$ on X and $f^m \mathcal{H}_V^k(\mathcal{F}^h) = 0$ on Q.

Let $\psi=\varphi\circ\cdots\circ\varphi$ (m times). Let $\mathscr{G}=\operatorname{Coker}\psi$. Then

$$S_{l+q}(\mathscr{G}\,|\,X-V)\subset S_{l+q+1}(\mathscr{F}\,|\,X-V).$$

Hence $\mathscr{H}_V^0(\mathscr{G}),\ldots,\mathscr{H}_V^q(\mathscr{G})$ and $\mathscr{H}_V^0(\mathscr{G}^h),\ldots,\mathscr{H}_V^q(\mathscr{G}^h)$ are coherent by Theorems A and A′.

The exact sequence $0\to\mathscr{F}\xrightarrow{\psi}\mathscr{F}\to\mathscr{G}\to 0$ yields the following commutative diagram with exact rows:

$$\begin{array}{ccccccc}
\mathscr{H}_V^{k-1}(\mathscr{F})^h & \longrightarrow & \mathscr{H}_V^{k-1}(\mathscr{G})^h & \longrightarrow & \mathscr{H}_V^{k}(\mathscr{F})^h & \xrightarrow{\psi_1} & \mathscr{H}_V^{k}(\mathscr{F})^h \\
\alpha\downarrow & & \beta\downarrow & & \gamma\downarrow & & \downarrow \\
\mathscr{H}_V^{k-1}(\mathscr{F}^h) & \longrightarrow & \mathscr{H}_V^{k-1}(\mathscr{G}^h) & \longrightarrow & \mathscr{H}_V^{k}(\mathscr{F}^h) & \xrightarrow{\psi_2} & \mathscr{H}_V^{k}(\mathscr{F}^h).
\end{array}$$

$\operatorname{Im}\psi_1=0$ on X and $\operatorname{Im}\psi_2=0$ on Q. By induction hypothesis, α and β are isomorphisms. Hence γ is isomorphic on Q. The induction process is complete, because Q is arbitrary. Q.E.D.

Finally, we give an interesting relation between the Ext functor and direct images in a very special case (Proposition 8 below).

LEMMA 9. *Suppose R is a ring, F is an R-module, and*

$$R^{p_{q+1}}\xrightarrow{\varphi_q}R^{p_q}\xrightarrow{\varphi_{q-1}}\cdots\xrightarrow{\varphi_1}R^{p_1}\xrightarrow{\varphi_0}R^{p_0}\xrightarrow{\varepsilon}F\longrightarrow 0$$

is an exact sequence. Suppose

$$0\xrightarrow{\varphi_{-1}^*}R^{p_0}\xrightarrow{\varphi_0^*}R^{p_1}\xrightarrow{\varphi_1^*}\cdots\xrightarrow{\varphi_{q-1}^*}R^{p_q}\xrightarrow{\varphi_q^*}R^{p_{q+1}}$$

is obtained by applying the functor $\operatorname{Hom}_R(\cdot,R)$ *to*

$$R^{p_{q+1}}\xrightarrow{\varphi_q}R^{p_q}\xrightarrow{\varphi_{q-1}}\cdots\xrightarrow{\varphi_1}R^{p_1}\xrightarrow{\varphi_0}R^{p_0}\longrightarrow 0.$$

Then $\operatorname{Ext}_R^k(F,R)\approx\operatorname{Ker}\varphi_k^*/\operatorname{Im}\varphi_{k-1}^*$ *for* $0\leqq k\leqq q$.

Proof. Fix $0\leqq k\leqq q$. We are going to prove $\operatorname{Ext}_R^k(F,R)\approx\operatorname{Ker}\varphi_k^*/\operatorname{Im}\varphi_{k-1}^*$ by induction on k. The case $k=0$ follows from the left-exactness of the functor $\operatorname{Hom}_R(\cdot,R)$.

Let $G=\operatorname{Im}\varphi_0$. The following commutative diagram has exact rows and exact columns:

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \uparrow & & & & & & \\
0 & \longrightarrow & G & \longrightarrow & R^{p_0} & \xrightarrow{\varepsilon} & F & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow\varphi_0 & & & & \\
 & & R^{p_1} & = & R^{p_1} & & & & \\
 & & \uparrow\varphi_1 & & & & & & \\
 & & R^{p_2}. & & & & & &
\end{array}$$

By applying the functor $\mathrm{Hom}_R(\cdot, R)$, we obtain the following commutative diagram with exact rows and exact columns:

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \downarrow & & \\ 0 = \mathrm{Ext}^1_R(R^{p_0}, R) & \longleftarrow & \mathrm{Ext}^1_R(F, R) & \longleftarrow & \mathrm{Hom}_R(G, R) & \longleftarrow & R^{p_0} \\ & & & & \downarrow & & \downarrow \varphi_0^* \\ & & & & R^{p_1} & = & R^{p_1} \\ & & & & \downarrow \varphi_1^* & & \\ & & & & R^{p_2}. & & \end{array}$$

We see readily that $\mathrm{Ext}^1_R(F, R) \approx \mathrm{Ker}\, \varphi_1^*/\mathrm{Im}\, \varphi_0^*$. The case $k=1$ is proved.

For the general case, assume $k>1$. The exact sequence $0 \to G \to R^{p_0} \to F \to 0$ yields the exact sequence $0=\mathrm{Ext}^{k-1}_R(R^{p_0}, R) \to \mathrm{Ext}^{k-1}_R(G, R) \to \mathrm{Ext}^k_R(F, R) \to \mathrm{Ext}^k_R(R^{p_0}, R)=0$. By induction hypothesis, $\mathrm{Ext}^{k-1}_R(G, R) \approx \mathrm{Ker}\, \varphi_k^*/\mathrm{Im}\, \varphi_{k-1}^*$. Hence the general case follows. Q.E.D.

PROPOSITION 8. *Suppose G is an open subset of C^n, V is an analytic subvariety of dimension r in G, and $\theta: G-V \to G$ is the inclusion map. Suppose $\mathscr{F}$ is a coherent analytic sheaf on G such that $\mathscr{F}$ is locally free on $G-V$. Then $\theta_k(\mathscr{F}|G-V) \approx \mathrm{Ext}^k_{\mathscr{H}}(\mathrm{Hom}_{\mathscr{H}}(\mathscr{F}, \mathscr{H}))$ for $0 \leqq k \leqq n-r-2$, where $\mathscr{H}$ is the analytic structure-sheaf of C^n.*

Proof. Fix $0 \leqq k \leqq n-r-2$. We are going to prove that $\theta_k(\mathscr{F}|G-V) \approx \mathrm{Ext}^k_{\mathscr{H}}(\mathrm{Hom}_{\mathscr{H}}(\mathscr{F}, \mathscr{H}))$. Let $\mathscr{G}=\mathrm{Hom}_{\mathscr{H}}(\mathscr{F}, \mathscr{H})$ and $\mathscr{G}^*=\mathrm{Hom}_{\mathscr{H}}(\mathscr{G}, \mathscr{H})$. Since the problem is local in nature, we can assume that we have the following exact sequence on G:

$$\mathscr{H}^{p_{n-r}} \xrightarrow{\varphi_{n-r-1}} \mathscr{H}^{p_{n-r-1}} \xrightarrow{\varphi_{n-r-2}} \cdots$$
$$\xrightarrow{\varphi_1} \mathscr{H}^{p_0} \xrightarrow{\varphi_0} \mathscr{H}^{p_0} \xrightarrow{\varepsilon} \mathscr{G} \longrightarrow 0.$$

By applying the functor $\mathrm{Hom}_{\mathscr{H}}(\cdot, \mathscr{H})$, we obtain

$$0 \longrightarrow \mathscr{G}^* \xrightarrow{\varepsilon^*} \mathscr{H}^{p_0} \xrightarrow{\varphi_0^*} \mathscr{H}^{p_1} \xrightarrow{\varphi_1^*} \cdots \tag{17}$$
$$\xrightarrow{\varphi_{n-r-2}^*} \mathscr{H}^{p_{n-r-1}} \xrightarrow{\varphi_{n-r-1}^*} \mathscr{H}^{p_{n-r}}.$$

Since $\mathscr{F}$ is locally free on $G-V$, (17) is exact on $G-V$ and $\mathscr{F} \approx \mathscr{G}^*$ on $G-V$. The following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 \longrightarrow & \theta_0(\mathscr{G}^*|G-V) & \longrightarrow & \theta_0(\mathscr{H}^{p_0}|G-V) & \longrightarrow & \theta_0(\mathscr{H}^{p_1}|G-V) \\ & & & \| & & \| \\ & & & \mathscr{H}^{p_0} & \xrightarrow{\varphi_0^*} & \mathscr{H}^{p_1} \end{array}$$

implies that $\theta_0(\mathscr{F}|G-V)\approx \operatorname{Ker}\varphi_0^*$. Hence the case $k=0$ follows from Lemma 9.

For the general case, assume $0<k\leqq n-r-2$. The exact sequence $0\to \operatorname{Ker}\varphi_\nu^* \to \mathscr{H}^{p_\nu}\to \operatorname{Ker}\varphi_{\nu+1}^*\to 0$ on $G-V$ yields the exact sequence $0=\theta_{k-\nu-1}(\mathscr{H}^{p_\nu}|G-V)\to \theta_{k-\nu-1}(\operatorname{Ker}\varphi_{\nu+1}^*|G-V)\to \theta_{k-\nu}(\operatorname{Ker}\varphi_\nu^*|G-V)\to \theta_{k-\nu}(\mathscr{H}^{p_\nu}|G-V)=0$ for $0\leqq \nu \leqq k-2$. Hence $\theta_k(\mathscr{F}|G-V)\approx\theta_k(\mathscr{G}^*|G-V)\approx\theta_k(\operatorname{Ker}\varphi_0^*|G-V)\approx\theta_{k-1}(\operatorname{Ker}\varphi_1^*|G-V)\approx\cdots\approx\theta_1(\operatorname{Ker}\varphi_{k-1}^*|G-V)$.

Since the following commutative diagram has exact rows and exact columns on $G-V$

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & \operatorname{Ker}\varphi_{k-1}^* & \longrightarrow & \mathscr{H}^{p_{k-1}} & \longrightarrow & \operatorname{Ker}\varphi_k^* & \longrightarrow & 0 \\
 & & & & \| & & \downarrow & & \\
 & & & & \mathscr{H}^{p_{k-1}} & \xrightarrow{\varphi_{k-1}^*} & \mathscr{H}^{p_k} & \xrightarrow{\varphi_k^*} & \mathscr{H}^{p_{k+1}} \\
 & & & & & & \downarrow & & \| \\
 & & & & 0 & \longrightarrow & \operatorname{Ker}\varphi_{k+1}^* & \longrightarrow & \mathscr{H}^{p_{k+1}} \\
 & & & & & & \downarrow & & \\
 & & & & & & 0 & &
\end{array}$$

the first and third rows and the columns of the following commutative diagram are exact

$$\begin{array}{ccccc}
 & & 0 & & \\
 & & \downarrow & & \\
\theta_0(\mathscr{H}^{p_{k-1}}|G-V) & \xrightarrow{\alpha} & \theta_0(\operatorname{Ker}\varphi_k^*|G-V) & \longrightarrow & \theta_1(\operatorname{Ker}\varphi_{k-1}^*|G-V) \\
 & & & & \longrightarrow \theta_1(\mathscr{H}^{p_{k-1}}|G-V)=0 \\
\| & & \downarrow & & \\
\theta_0(\mathscr{H}^{p_{k-1}}|G-V) & \xrightarrow{\tilde\varphi_{k-1}} & \theta_0(\mathscr{H}^{p_k}|G-V) & \xrightarrow{\tilde\varphi_k} & \theta_0(\mathscr{H}^{p_{k+1}}|G-V) \\
 & & \downarrow & & \| \\
0 & \longrightarrow & \theta_0(\operatorname{Ker}\varphi_{k+1}^*|G-V) & \longrightarrow & \theta_0(\mathscr{H}^{p_{k+1}}|G-V).
\end{array}$$

Hence $\theta_0(\operatorname{Ker}\varphi_k^*|G-V)\approx\operatorname{Ker}\tilde\varphi_k$, $\operatorname{Im}\alpha\approx\operatorname{Im}\tilde\varphi_{k-1}$, and

$$\theta_1(\operatorname{Ker}\varphi_{k-1}^*|G-V)\approx\theta_0(\operatorname{Ker}\varphi_k^*|G-V)/\operatorname{Im}\alpha.$$

Since $\theta_0(\mathscr{H}^{p_\nu}|G-V)\approx\mathscr{H}^{p_\nu}$ for $\nu=k-1, k, k+1$, we have $\tilde\varphi_\nu=\varphi_\nu^*$ for $\nu=k-1, k$. Therefore $\theta_k(\mathscr{F}|G-V)\approx\theta_1(\operatorname{Ker}\varphi_{k-1}^*|G-V)\approx\operatorname{Ker}\tilde\varphi_k/\operatorname{Im}\tilde\varphi_{k-1}=\operatorname{Ker}\varphi_k^*/\operatorname{Im}\varphi_{k-1}^*$. By Lemma 9, $\operatorname{Ker}\varphi_k^*/\operatorname{Im}\varphi_{k-1}^*\approx\operatorname{Ext}^k_{\mathscr{H}}(\mathscr{G},\mathscr{H})=\operatorname{Ext}^k_{\mathscr{H}}(\operatorname{Hom}_{\mathscr{H}}(\mathscr{F},\mathscr{H}),\mathscr{H})$. Q.E.D.

After the submission of this paper, I learned that G. Trautmann simultaneously and independently had also obtained the equivalence of (i) and (iii) of Theorem A in Invent. Math. **8** (1969), 143–174.

REFERENCES

1. J. Frisch and J. Guenot, *Prolongement de faisceaux analytiques cohérents*, Invent. Math. **7** (1969), 321–343.

2. A. Grothendieck, *Cohomologie locale des faisceaux cohérents et théorèmes de Lefschetz locaux et globaux*, Séminaire de Géométrie Algébrique, Inst. Hautes Études Sci., Paris, 1962. MR **35** #1604.

3. R. Hartshome, *Local cohomology*, Springer-Verlag, Berlin, 1967. MR **37** #219.

4. M. Nagata, *Local rings*, Interscience, New York, 1962. MR **27** #5790.

5. G. Scheja, *Riemannsche Hebbarkeitssätze für Cohomologieklassen*, Math. Ann. **144** (1961), 345–360. MR **26** #6437.

6. ———, *Fortsetzungssätze der komplex-analytischen Cohomologie und ihre algebraische Charakterisierung*, Math. Ann. **157** (1964), 75–94. MR **31** #738.

7. J.-P. Serre, *Faisceaux algébriques cohérents*, Ann. of Math. **157** (1964), 197–278.

8. ———, *Géométrie algébrique et géométrie analytique*, Ann. Inst. Fourier Grenoble **6** (1955/56), 1–42. MR **18**, 511.

9. ———, *Prolongement de faisceaux analytiques cohérents*, Ann. Inst. Fourier Grenoble **16** (1966), fasc. 1, 363–374. MR **35** #3088.

10. Y.-T. Siu, *Noether-Lasker decomposition of coherent analytic subsheaves*, Trans. Amer. Math. Soc. **135** (1969), 375–385. MR **38** #2340.

11. ———, *Extension of locally free analytic sheaves*, Math. Ann. **179** (1969), 285–294.

12. ———, *Absolute gap-sheaves and extensions of coherent analytic sheaves*, Trans. Amer. Math. Soc. **141** (1969), 361–376.

13. ———, *Extending coherent analytic sheaves through subvarieties*, Bull. Amer. Math. Soc. **75** (1969), 123–126.

14. ———, *Extending coherent analytic sheaves*, Ann. of Math. **90** (1969), 108–143.

15. ———, *Analytic sheaves of local cohomology*, Bull. Amer. Math. Soc. **75** (1969), 1011–1012.

16. W. Thimm, *Lückengarben von kohärenten analytischen Modulgarben*, Math. Ann. **148** (1962), 372–394. MR **26** #1493.

17. ———, *Struktur- und Singularitätsuntersuchungen an kohärenten analytischen Modulgarben*, J. Reine Angew. Math. **234** (1969), 123–151.

18. G. Trautmann, *Ein Kontinuitätssatz für die Fortsetzung kohärenter analytischer Garben*, Arch. Math. **18** (1967), 188–196. MR **35** #6869.

19. ———, *Abgeschlossenheit von Corandmoduln und Fortsetzbarkeit kohärenter analytischer Garben*, Invent. Math. **4** (1968), 216–230. MR **37** #4289.

20. ———, *Fortsetzung lokal-freier Garben über 1-dimensionale Singularitätenmengen*, Ann. Scuola Norm. Sup. Pisa **23** (1969), 155–184.

21. ———, *Cohérence des faisceaux analytiques de la cohomologie local*, C. R. Acad. Sci. Paris **267** (1968), 694–695.

22. O. Zariski and P. Samuel, *Commutative algebra*. Vols. I, II, Van Nostrand, Princeton, N. J., 1958, 1960. MR **19**, 833. MR **22** #11006.

UNIVERSITY OF NOTRE DAME,
NOTRE DAME, INDIANA 46556